

Nonorientable 3-manifolds admitting coloured triangulations with at most 30 tetrahedra *

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Abstract

We present the census of all non-orientable, closed, connected 3-manifolds admitting a rigid crystallization with at most 30 vertices. In order to obtain the above result, we generate, manipulate and compare, by suitable computer procedures, all rigid non-bipartite crystallizations up to 30 vertices.

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1 Introduction

Within the study of 3-dimensional manifolds, it is often useful to have significative examples to formulate and test conjectures, to obtain classification results or to investigate patterns for 3-manifolds.

During the last ten years, several papers have been published containing tables (*censuses*) of 3-manifolds, satisfying certain conditions. The criterion, which is usually adopted, is to bound the possible number of tetrahedra in a triangulation of the manifold. First Matveev presented the census of closed orientable irreducible 3-manifolds having a triangulation formed by at most six tetrahedra ([24]). More precisely, Matveev's results are based on the representation of 3-manifolds by special spines and his bound is the complexity of the manifold, i.e. the minimal number of vertices in a special spines of the manifold, which coincides (excluding some very particular cases) with the number of tetrahedra in a minimal triangulation.

The orientable censuses were later extended by Ovchinnikov up to complexity 7 ([28]), by Martelli and Petronio up to complexity 10 ([22], [23]) and by Matveev himself up to complexity 11 ([26], [27]).

With regard to the non-orientable case, the first tables were made by Amendola and Martelli up to complexity 7 ([1], [2]) and Burton up to 7 tetrahedra ([5], [6]); recently Burton completed the census up to 10 tetrahedra ([7]).

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In this paper, we share Burton's approach of listing and analyzing all possible triangulations of closed 3-manifolds, restricting our considerations to "coloured triangulations" or, equivalently, to their dual "edge-coloured graphs" (see [16], [3]).

Edge-coloured graphs can be easily encoded by matrices and thus manipulated by computer, in order to recognize topological properties and compute invariants of the underlying manifolds or in order to change triangulations by means of moves which preserve the homeomorphism type of the represented manifolds.

Within the theory of edge-coloured graphs, several results have been obtained in generating and classifying catalogues of closed 3-manifolds.

The adopted bounds are usually the number of vertices of the graph (equivalently the number of tetrahedra of the coloured triangulation) or the *regular genus* of the graph, an invariant whose minimal value coincides with the Heegaard genus of the represented manifold.

In the first case, orientable catalogues were first produced and analyzed by Lins up to 28 vertices ([21]; the classification was completed in [8]); later they were extended to 30 vertices in [12]. Moreover, Casali in [9] started the generation and study of non-orientable catalogues, completing it up to 26 vertices.

In this paper, we extend the above result to the cases of non-orientable closed manifolds representable by edge-coloured graphs with 28 and 30 vertices. The generation procedure remains as in [9], while the main point in the classification is an algorithm, already introduced in [12], for subdividing the catalogues into classes so that the elements of each class represent the same manifold.

As further step, we identified the manifolds represented by each class, by computation of invariants, comparison with known edge-coloured graphs and, in some cases, by constructing coloured triangulations of manifolds in Burton's tables which matched our representatives.

Finally, we found that there exist exactly thirty-three closed non-orientable 3-manifolds, of which sixteen are prime, admitting a coloured triangulation with at most 30 tetrahedra. A precise description of the above prime manifolds will be presented at the end of section 4.

Existing catalogues of genus two orientable manifolds have been generated and studied up to 34 ([11],[4]) and 42 tetrahedra ([19]). Presently, we are examining the non-orientable case up to 42 tetrahedra: the related results will be the subject of a forthcoming paper.

2 Coloured triangulations of 3-manifolds

Throughout this paper, manifolds, when not otherwise specified, will always be closed and connected.

A coloured n -complex is a pseudocomplex ([18]) K of dimension n with a labelling of its vertices by $\Delta_n = \{0, \dots, n\}$, which is injective on the vertex-set of each simplex of K .

An $(n+1)$ -coloured graph is a pair (Γ, γ) , where Γ is a graph, regular of degree $n+1$, and $\gamma: E(\Gamma) \rightarrow \Delta_n$ a map which is injective on each pair of adjacent edges of Γ .

In the following, we shall often write Γ instead of (Γ, γ) .

For each $B \subseteq \{0, \dots, n\}$, we call B -residues of (Γ, γ) the connected components of the coloured graph $\Gamma_B = (V(\Gamma), \gamma^{-1}(B))$; given an integer $m \in \{1, \dots, n\}$ we call m -residue of Γ

each B -residue of Γ with $\#B = m$.

An isomorphism $\phi : \Gamma \rightarrow \Gamma'$ is called a *coloured isomorphism* between the $(n+1)$ -coloured graphs (Γ, γ) and (Γ', γ') if there exists a permutation φ of Δ_n such that $\varphi \circ \gamma = \gamma' \circ \phi$.

Coloured graphs are an useful tool for representing manifolds (see [3] and [16] for a survey on this topic); in fact there is a bijective correspondence between a particular class of $(n+1)$ -coloured graphs and the class of coloured triangulations of n -manifolds.

A direct way to see this correspondence is to consider, for each $(n+1)$ -coloured graph Γ , the coloured complex $K(\Gamma)$ obtaining by the following rule:

- for each vertex v of Γ , take an n -simplex $\sigma(v)$ and label its vertices by Δ_n ;
- if v and w are vertices of Γ joined by an i -coloured edge ($i \in \Delta_n$), then identify the $(n-1)$ -faces of $\sigma(v)$ and $\sigma(w)$ opposite to the i -coloured vertex.

See [16] for a more precise description of the involved constructions.

If M is a manifold of dimension n and Γ an $(n+1)$ -coloured graph such that $|K(\Gamma)| \cong M$, we say that M is *represented* by Γ .

If, for each $i \in \Delta_n$, Γ_i is connected (equivalently the corresponding coloured triangulation $K(\Gamma)$ has exactly one i -coloured vertex for each $i \in \Delta_n$), then both the $(n+1)$ -coloured graph Γ and the coloured triangulation $K(\Gamma)$ are called *contracted*; furthermore, if Γ represents an n -manifold M , then it is called a *crystallization* of M . Note that M is orientable iff Γ is bipartite.

We can construct a coloured graph representing the connected sum of two n -manifolds M' and M'' starting from their graphs. In fact let Γ' and Γ'' be $(n+1)$ -coloured graphs representing M' and M'' respectively. Let x be a vertex of Γ' and y a vertex of Γ'' , then the $(n+1)$ -coloured graph $\Gamma = \Gamma' \# \Gamma''$ obtained by removing x from Γ' and y from Γ'' and by gluing the "hanging" edges according to their colours, represents $M' \# M''$ (see [16])¹.

Remark 1 If Γ is a $(n+1)$ -coloured graph representing a n -manifold M and if there are in Γ $n+1$ edges $\{e_0, \dots, e_n\}$, one for each colour $i \in \Delta_n$, such that $\Gamma - \{e_0, \dots, e_n\}$ splits into two connected components, then it is easy to reverse the above procedure and construct two $(n+1)$ -coloured graphs Γ' and Γ'' , representing two n -manifolds M' and M'' respectively, such that $\Gamma = \Gamma' \# \Gamma''$, hence $M = M' \# M''$.

An important role, within the theory of coloured graphs, is played by combinatorial moves (*dipole moves*) which transform an $(n+1)$ -coloured graph representing an n -manifold into another (usually non-colour isomorphic) $(n+1)$ -coloured graph, representing the same manifold.

If x, y are two vertices of a $(n+1)$ -coloured graph (Γ, γ) joined by k edges $\{e_1, \dots, e_k\}$ with $\gamma(e_h) = i_h$, for $h = 1, \dots, k$, then we call $\theta = \{x, y\}$ a k -*dipole* or a *dipole of type k* in Γ , *involving colours i_1, \dots, i_k* , iff x and y belong to different $(\Delta_n - \{i_1, \dots, i_k\})$ -residues of Γ .

In this case a new $(n+1)$ -coloured graph (Γ', γ') can be obtained from Γ by deleting x, y and all their incident edges and joining, for each $i \in \Delta_n - \{i_1, \dots, i_k\}$, the vertex i -adjacent

¹It is well-known that, if both manifolds don't admit orientation-preserving automorphisms, there exist two non-homeomorphic connected sums. Each corresponds to requiring x to belong to a fixed bipartition class in $V(\Gamma')$ and choosing y in one of the two different bipartition classes of $V(\Gamma'')$

to x to the vertex i -adjacent to y ; (Γ', γ') is said to be obtained from (Γ, γ) by *deleting the k -dipole θ* . Conversely (Γ, γ) is said to be obtained from (Γ', γ') by *adding the k -dipole*.

By restricting ourselves to 3-manifolds (in the following this will always be the case), we can introduce further moves.

Let (Γ, γ) be a 4-coloured graph. Let Θ be a subgraph of Γ formed by a $\{i, j\}$ -coloured cycle C of length $m+1$ and a $\{h, k\}$ -coloured cycle C' of length $n+1$, having only one common vertex x_0 and such that $\{i, j, h, k\} = \{0, 1, 2, 3\}$. Then Θ is called an (m, n) -dipole.

If x_1, x_m, y_1, y_n are the vertices respectively i, j, h, k -adjacent to x_0 , we define the 4-coloured graph (Γ', γ') obtained from Γ by *cancelling the (m, n) -dipole*, in the following way:

- 1) delete Θ from Γ and consider the product Ξ of the subgraphs $C - \{x_0\}$ and $C' - \{x_0\}$;
- 2) for each $s, s' \in \{1, \dots, n\}$ (resp. for each $r, r' \in \{1, \dots, m\}$), let e be the edge joining y_s and $y_{s'}$ (resp. x_r and $x_{r'}$) in Γ . If $\gamma(e) = c \in \{0, 1, 2, 3\}$, then, for each $t \in \{1, \dots, m\}$ (resp. for each $t \in \{1, \dots, n\}$), join the vertices (x_t, y_s) and $(x_t, y_{s'})$ (resp. (x_r, y_t) and $(x_{r'}, y_t)$) by a c -coloured edge in Ξ ;
- 3) for all $r \in \{1, \dots, m\}$, $s \in \{1, \dots, n\}$, if a vertex z of $\Gamma - \Theta$ is joined to y_s (resp. x_r) by a i or j (resp. h or k)-coloured edge in Γ , then z is joined to (x_1, y_s) , (x_m, y_s) (resp. (x_r, y_1) , (x_r, y_n)) by a i or j (resp. h or k)-coloured edge in Γ' .

Figure 2.1 shows the whole process in the case $m = 3$ and $n = 5$.

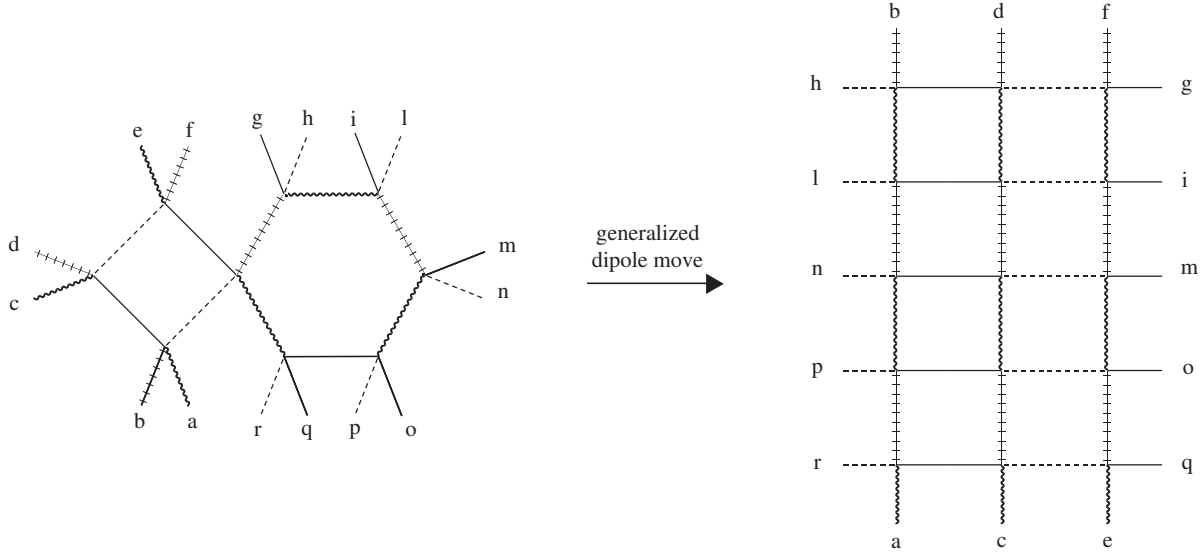


Figure 2.1

The moves just described are called *generalized dipole moves*. We can summarize in the following result the significance of dipole moves and generalized dipole moves as a tool to manipulate 4-coloured graphs.

Proposition 1 ([15]) *If (Γ, γ) and (Γ', γ') are 4-coloured graph representing two 3-manifolds M and M' respectively, and (Γ', γ') is obtained from (Γ, γ) by a dipole move or a generalized dipole move, then $M \cong M'$.*

Moreover there exist further moves, which can be applied to each 4-coloured graph Γ , so to eliminate particular configurations.

If two i -coloured edges $e, f \in E(\Gamma)$ belong to the same $\{i, j\}$ -coloured cycle and to the same $\{i, k\}$ -coloured cycle of Γ , with $j, k \in \Delta_3 - \{i\}$ (resp. to the same $\{i, h\}$ -coloured cycle of Γ , for each $h \in \Delta_3 - \{i\}$), then (e, f) is called a ρ_2 -pair (resp. a ρ_3 -pair). Usually, we will write ρ -pair instead of ρ_2 -pair or ρ_3 -pair.

The graph Γ is a *rigid crystallization* of a 3-manifold M^3 if it is a crystallization of M^3 and contains no ρ -pairs.

A non-rigid crystallization Γ of a 3-manifold M can always be transformed into a rigid one by *switching ρ -pairs* (see [21]) and cancelling the dipoles which could be created in the process. The switching of a ρ_2 -pair doesn't change the represented manifold, while, for a ρ_3 -pair, we have the following result.

Lemma 2 ([21]) *Let Γ be a 4-coloured graph containing a ρ_3 -pair, if Γ' , obtained from Γ by switching it, is a crystallization of a 3-manifold M , then Γ represents the 3-manifold $M \# H$, where $H = \mathbb{S}^1 \times \mathbb{S}^2$ iff Γ and Γ' are both bipartite or both non bipartite, otherwise $H = \mathbb{S}^1 \tilde{\times} \mathbb{S}^2$.*

Since each closed connected 3-manifold admits a rigid crystallization (see [8] for a detailed proof), we can always require the rigidity condition to be satisfied with no loss with regard to the represented manifolds.

An essential tool to deal with coloured graphs by computer is the *code*, that is a numerical "string", which describes completely the combinatorial structure of the coloured graph (see [14] for definition and description of the related *rooted numbering algorithm*). More precisely, we can state

Lemma 3 ([14]) *Two $(n + 1)$ -coloured graphs are colour-isomorphic iff they have the same code.*

Therefore, by representing each coloured graph by its code, we can easily reduced any catalogue of crystallizations to one containing only non-colour-isomorphic graphs.

A different description of triangulations for 3-manifolds, including the coloured ones, can be found in [5], [6], by means of face pairings and gluing permutation selections; moreover, each triangulation has its dual skeleton as an associated 4-valent graph. For a fixed triangulation of a 3-manifold the face pairing is intrinsic in the definition of the triangulation and it is called the *associated face pairing*. In particular a coloured triangulation is a triangulation in the sense of [5], [6], equipped with the associated face pairing and having the identity map as gluing permutation selection. The associated graph, if coloured by associating to each edge the colour of the opposite vertex, is exactly the 4-coloured graph representing the coloured triangulation.

3 Generating and analysing catalogue $\tilde{\mathcal{C}}^{(30)}$

In this section we will describe the generation and analysis of the catalogue $\tilde{\mathcal{C}}^{(30)}$ of all non-isomorphic rigid non-bipartite crystallizations with at most 30 vertices. Since the rigidity condition is not restrictive with regard to the represented manifolds, the topological classification of the crystallizations in $\tilde{\mathcal{C}}^{(30)}$ yields the list of all non-orientable closed 3-manifolds admitting a coloured triangulation with at most 30 tetrahedra; moreover the catalogue $\tilde{\mathcal{C}}^{(30)}$ also yields the list of such triangulations encoded into graphs.

In [8] an algorithm is introduced, which for each $p \in \mathbb{N}$, produces the archive $\mathcal{C}^{(2p)}$ (resp. $\tilde{\mathcal{C}}^{(2p)}$) of codes of all non-isomorphic rigid bipartite (resp. non-bipartite) crystallizations with exactly $2p$ vertices.

For our convenience, we summarize below the main steps of the algorithm.

- Step 1: By induction on p and by making use of the results of [20] and [21], we construct the set $\mathcal{S}^{(2p)} = \{\Sigma_1^{(2p)}, \Sigma_2^{(2p)}, \dots, \Sigma_{n_p}^{(2p)}\}$ of all (connected) rigid 3-coloured graphs with $2p$ vertices representing \mathbb{S}^2 .
- Step 2: For each $i = 1, 2, \dots, n_p$, we add p edges coloured by 3 to $\Sigma_i^{(2p)}$ in all ways so to obtain 4-coloured graphs, provided that the planarity of the \hat{i} -residues ($i \in \{0, 1, 2\}$) and the rigidity of the whole graph are preserved after each edge is added. Each time a regular 4-coloured graph is obtained, we check whether it is a crystallization (i.e. its Euler characteristic is zero).
- Step 3: By comparing the codes and by checking rigidity condition and bipartition property on the crystallizations arising from Steps 1 and 2, we form the catalogue $\mathcal{C}^{(2p)}$ (resp. $\tilde{\mathcal{C}}^{(2p)}$) of all rigid bipartite (resp. non-bipartite) crystallizations with $2p$ vertices.

With regard to the non-bipartite case, the output data of a C++ program implementing the above algorithm, are shown in the following Table (see [12] for the orientable case).

2p	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$\#\tilde{\mathcal{C}}^{(2p)}$	0	0	0	0	0	0	1	1	1	9	12	88	480	2790	21804

Table 1: non-bipartite rigid crystallizations up to 30 vertices.

Catalogues $\tilde{\mathcal{C}}^{(2p)}$, for $p < 14$, have been analysed and the represented manifolds identified in [8] (see also [13] and [12] for the orientable case), mainly by manipulating the crystallizations through generalized dipole moves and by subdividing them into classes according to the equivalence defined by the moves.

In this paper we follow the same idea and generalize it into a more refined “classification” algorithm: more precisely, we will show how to subdivide a given list X of rigid crystallizations into disjoint classes $\{c_1, \dots, c_s\}$ such that, for each $i \in \{1, \dots, s\}$ and for each $\Gamma, \Gamma' \in c_i$, there exist two integers $h, k \geq 0$ and a 3-manifold M such that $|K(\Gamma)| = M \#_h H$ and $|K(\Gamma')| =$

$M \#_k H$, where $\#_r H$ denotes the connected sum of r copies either of the orientable or of the non-orientable \mathbb{S}^2 -bundle over \mathbb{S}^1 ; more precisely $H = \mathbb{S}^1 \times \mathbb{S}^2$ iff Γ and Γ' are both bipartite or both non-bipartite.

To make the algorithm clearer, let us introduce some definitions and notations.

Let Γ be a rigid crystallization and suppose an ordering of its vertices is fixed so that we can write $V(\Gamma) = \{v_1, \dots, v_{2p}\}$; given an integer $i \in \{1, 2, 3\}$, we denote by $\theta_i(\Gamma)$ the rigid crystallization obtained from Γ by subsequent cancellations of (m, n) -dipoles of type $\{0, i\}$, according to the following rules:

- $m, n < 9$ (this condition is necessary in order to bound the possible number of vertices of $\theta_i(\Gamma)$).
- Generalized dipoles of type $\{0, i\}$ are looked for and cancelled for increasing value of the integer $m \cdot n$ and by starting from vertex v_1 up to v_{2p} , i.e if $\delta(v_i)$ is a (m, n) -generalized dipole at vertex v_i (resp. $\delta'(v_j)$ is a (m', n') -generalized dipole at vertex v_j), then the cancellation of $\delta(v_i)$ is performed before the cancellation of $\delta(v_j)$ iff $(m \cdot n < m' \cdot n')$ or $(m \cdot n = m' \cdot n' \text{ and } i < j)$.
- After each generalized dipole cancellation, proper dipoles and ρ -pairs are cancelled in the resulting graph.

Moreover, we define $\theta_0(\Gamma) = \Gamma$.

Given a rigid crystallization Γ , there is a natural way to construct a rigid crystallization $\Gamma^<$ which is colour-isomorphic to Γ and such that an ordering is induced in $V(\Gamma^<)$ by the *rooted numbering algorithm* generating the code of Γ (see [14]). As a consequence, for each $i \in \{0, 1, 2, 3\}$, we can define a map θ_i on any set X of rigid crystallizations by setting, for each $\Gamma \in X$, $\theta_i(\Gamma) = \theta_i(\Gamma^<)$, with the ordering of the vertices induced by the code of Γ .

Let us denote by S_3^0 the set of all permutations on Δ_3 , which fix the element 0. If S_3^0 is considered as a lexicographically ordered set, let $\delta^{(k)} = (\delta_0^{(k)} = 0, \delta_1^{(k)}, \delta_2^{(k)}, \delta_3^{(k)})$ ($k \in \{1, 2, \dots, 6\}$) denote the k -th element of S_3^0 .

For each $k \in \{1, 2, \dots, 6\}$ and for each $i \in \Delta_3$, we set

$$\ll \delta_i^{(k)} \gg = \theta_{\delta_i^{(k)}} \circ \theta_{\delta_{i-1}^{(k)}} \circ \dots \circ \theta_{\delta_0^{(k)}}.$$

Let us now consider the following set of moves:

$$\begin{aligned} \bar{\mathcal{S}} = & \{ \ll \delta_i^{(k)} \gg \mid k \in \{1, 2, \dots, 6\} \text{ and } i \in \Delta_3 \} \cup \\ & \cup \{ \ll \delta_i^{(k)} \gg \circ \ll \delta_3^{(k-1)} \gg \circ \dots \circ \ll \delta_3^{(1)} \gg \mid k \in \{2, \dots, 6\} \text{ and } i \in \Delta_3 \}, \end{aligned}$$

and, for each rigid crystallization Γ and for each $\epsilon \in \bar{\mathcal{S}}$, let $\theta_\epsilon(\Gamma)$ be the rigid crystallization obtained by applying the sequence of moves ϵ to Γ .

Note that, by Propositions 1 and 2, each sequence of moves $\epsilon \in \bar{\mathcal{S}}$ transforms a rigid crystallization of a 3-manifold M into a rigid crystallization of a 3-manifold M' , such that

$M = M' \#_t H$ (H as above) and t is the number of ρ_3 -pairs, which have been deleted while performing the sequence ϵ . As a consequence, the following definitions naturally arise.

For each $\Gamma \in X$, the class $cl(\Gamma)$ of Γ is defined as

$$cl(\Gamma) = \{\Gamma' \in X \mid \exists \epsilon, \epsilon' \in \bar{\mathcal{S}} \text{ s.t.} \\ \theta_\epsilon(\Gamma) \text{ and } \theta_{\epsilon'}(\Gamma') \text{ have the same code}\}.$$

Furthermore, we will denote by $h_\epsilon(\Gamma)$ the number of ρ_3 -pairs which have been deleted by passing from Γ to $\theta_\epsilon(\Gamma)$ (obviously it could be zero).

Let us describe now the algorithm which, starting from an ordered list X of rigid crystallizations, simultaneously produces, for each $\Gamma \in X$, the set $cl(\Gamma)$ and a non-negative number $h(\Gamma)$, whose meaning will be clear in the following.

More precisely, we will form $cl(\Gamma)$ and compute $h(\Gamma)$ in the following way.

Step 1: We set $cl(\Gamma) = \{\Gamma\}$ and $h(\Gamma) = 0$;

Step 2: for each $\epsilon \in \bar{\mathcal{S}}$, if there exist $\Gamma' \in X$ (coming before Γ in X) and $\epsilon' \in \bar{\mathcal{S}}$ such that the codes of $\theta_\epsilon(\Gamma)$ and $\theta_{\epsilon'}(\Gamma')$ coincide, then

- if $h(\Gamma') - h_{\epsilon'}(\Gamma') \geq h(\Gamma) - h_\epsilon(\Gamma)$, set $h(\Gamma'') = k - h(\Gamma) + h_\epsilon(\Gamma) + h(\Gamma') - h_{\epsilon'}(\Gamma')$ for each $\Gamma'' \in cl(\Gamma)$ with $h(\Gamma'') = k$;
- if $h(\Gamma') - h_{\epsilon'}(\Gamma') < h(\Gamma) - h_\epsilon(\Gamma)$, set $h(\Gamma'') = k + h(\Gamma) - h_\epsilon(\Gamma) - h(\Gamma') + h_{\epsilon'}(\Gamma')$ for each $\Gamma'' \in cl(\Gamma')$ with $h(\Gamma'') = k$;

In both cases, set $c = cl(\Gamma) \cup cl(\Gamma')$ and $cl(\Gamma'') = c$, for each $\Gamma'' \in c$.

Furthermore, for each $c_i = \{\Gamma_1^i, \dots, \Gamma_{r_i}^i\}$ and for each $0 \leq h \leq \max\{h(\Gamma_1^i), \dots, h(\Gamma_{r_i}^i)\}$, the class c_i can be naturally subdivided into subsets $c_{i,h} = \{\Gamma_j^i \in c_i \mid h(\Gamma_j^i) = h\}$.

By Propositions 1 and 2, it follows very easily that, if $\Gamma \in X$, Γ bipartite (resp. non-bipartite), represents the manifold M with $h(\Gamma) = h$ and $c_i = cl(\Gamma)$, then each element of $c_{i,k}$ ($0 \leq k \leq \max\{h(\Gamma') \mid \Gamma' \in c_i\}$) represents the manifold M' with $M' = M \#_{k-h} H$ or $M = M' \#_{h-k} H$ according to $k \geq h$ or $k < h$ and H as above.

Remark 2 Note that the algorithm works as well for any set \mathcal{S} of sequences of generalized dipoles moves, dipole moves and ρ -pairs switching, provided that each element of \mathcal{S} transforms rigid crystallizations into rigid crystallizations; actually we could have described the above procedure for such a general set \mathcal{S} independently from how the moves were performed (for this approach see [13]). However, we preferred to restrict ourselves to the particular set $\bar{\mathcal{S}}$, which was used for our implementation.

It is clear that, if there exist $i \in \{1, \dots, s\}$ and $\Gamma \in c_i$ such that $|K(\Gamma)|$ is known, then all manifolds represented by crystallizations of c_i are completely identified.

Therefore, if known catalogues of crystallizations are inserted in X , all classes of X containing at least one known crystallization are completely identified.

According to this idea the classification algorithm has been implemented in the C++ program Γ -class²: its input data are a list X of rigid crystallizations and the informations about the already known crystallizations of X (possibly none), i.e the identification of their represented manifolds through suitable “names”; the output is the list of classes of X , together with their representatives and, if possible, their names.

We have applied Γ -class to $X = \tilde{\mathcal{C}}^{(30)} = \bigcup_{1 \leq p \leq 15} \mathcal{C}^{(2p)}$, obtaining 32 classes: twelve contained crystallizations of $\tilde{\mathcal{C}}^{(2p)}$ with $p < 14$ and were therefore completely identified. Furthermore four contained non-orientable handles and were recognized by means of switching of ρ_3 -pairs and comparison with catalogues $\mathcal{C}^{(2p)}$, $1 \leq p \leq 15$ of rigid bipartite crystallizations.

Our further step was to apply Remark 1 to all crystallizations of the unknown 16 classes, in order to recognize possible connected sums: more precisely it was necessary to check the condition of Remark 1 on each crystallization Γ of an unknown class and, in case of it being satisfied, to construct the crystallizations Γ' and Γ'' and try their recognition. This has also been made by program Γ -class, and the results involved 7 classes. They all represented distinct manifolds.

4 Main results

Before presenting our results with regard to the complete identification of the manifolds, whose minimal (with regard to the number of vertices) rigid crystallizations have 28 or 30 vertices, let us introduce some notations.

We will denote by $TB(A)$ the torus bundle corresponding to the matrix $A \in GL(2, \mathbb{Z})$, i.e. the closed 3-manifold obtained as quotient of $T \times I$, by identifying the bottom and top torus by the homeomorphism of T^2 , induced by the matrix A (with respect to a fixed basis of $\pi_1(T^2)$). Recall that $TB(A)$ is non-orientable if and only if $\det A = -1$.

In [9] a construction is described to obtain a 4-coloured graph representing $TB(A)$ for each $A \in GL(2, \mathbb{Z})$. Figures 4.1 and 4.2 sketch the whole process for the manifold $TB \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$. In particular, Figure 4.1 shows a subdivision of the boundary tori of $T \times I$ with the curves c_i and c'_i ($i = 0, 1$), whose identification produces the self-homeomorphism of T defined by the matrix A .

Figure 4.2 shows the boundary of a 3-dimensional simplicial complex K' , which is the first barycentric subdivision of the cube K representing $I \times I \times I$, triangulated by subdividing the

² Γ -class has been developed by M.R. Casali and P. Cristofori and is available at WEB page <http://cdm.unimo.it/home/matematica/casali.mariarita/CATALOGUES.htm> where a detailed description of the program can be found, too

top and bottom squares as in Figure 4.1 and performing the join on the boundary of K from one vertex in its interior.

K' is coloured by labelling i , for each $i = 0, 1, 2, 3$, the vertices dual to the i -dimensional simplices of K . Finally the coloured triangulation of $TB(A)$ is obtained by identifying, for each $h = 1, \dots, 80$, the pairs of triangles on $\partial K'$, which in the Figure are labelled h and h' respectively; the identification is obviously performed by respecting the colouring of the vertices.

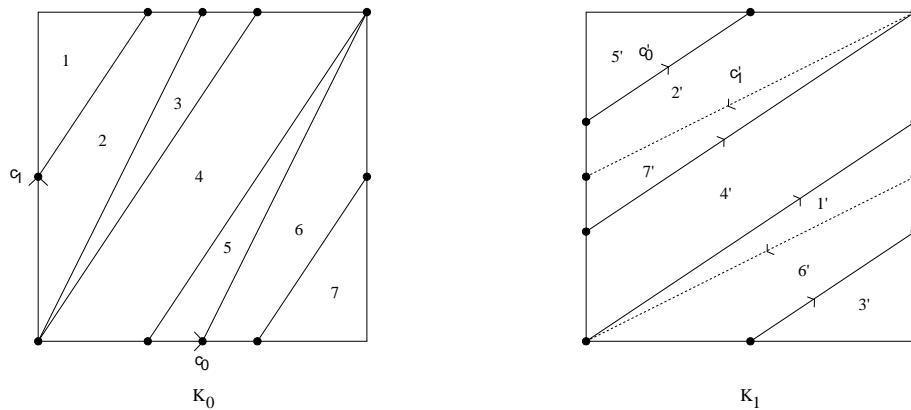


Figure 4.1

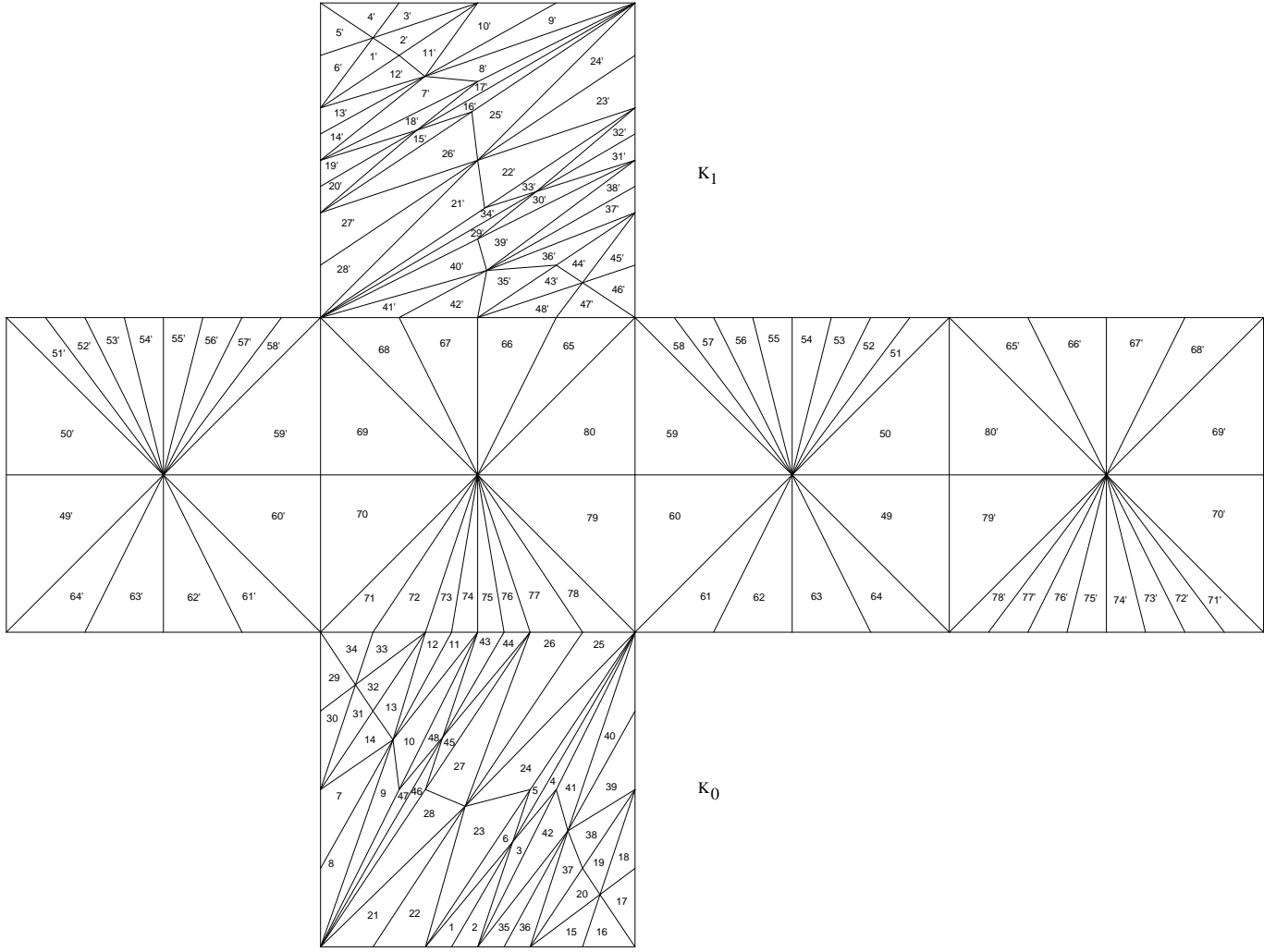


Figure 4.2

Moreover, $SFS(S, (\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r))$ will denote the Seifert fibred space with base space the orbifold S and (non-normalized) parameters $(\alpha_1, \beta_1), \dots, (\alpha_r, \beta_r)$ (in our case, we always have $r \leq 2$). The base orbifolds of the Seifert manifolds in our catalogue, are

\mathbb{RP}^2 , projective plane

\overline{D} , disc with reflector boundary

T^2/o_2 , torus containing fibre-reversing curves

K^2 , Klein bottle

K^2/n_3 , Klein bottle containing fibre-reversing curves with non-orientable total space

A , annulus.

Furthermore $SFS(A, (2, 1)) \cup SFS(A, (2, 1))/N$ is the non-geometric graph-manifold obtained by pasting together two copies of $SFS(A, (2, 1))$ along their boundary tori according to the matrix N (which in our case is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$).

By the results of program Γ -class, we identified 23 classes: there remain nine unknown ones. Here is the list of codes of their representatives, which are the first to appear in our catalogue.

- $\Gamma^{(1)}$ CABFDEIGHLJKNMINDCMGFJLHEAKBJhKnHljbDgCfLdGEkiBMAeNacmFI
- $\Gamma^{(2)}$ DABCHEFGKIJMLONKNFEDCBIHLOJGMAGliNOkADcofbKHjgLIhJManCEmBFd
- $\Gamma^{(3)}$ DABCGEFJHIMKLONJNLEDCHGKOIFBMAMieKcJJobFDOAchmGnkNjBLgfdHaEl
- $\Gamma^{(4)}$ CABFDEIGHLJKNMIMDCKGFJNHEBLAMIFBHjNIDfnhAkmdJiLcKebCGEag
- $\Gamma^{(5)}$ EABCDIFGHLJKOMNLONGFEDCJIMAKHBMkIHNBIDCmbgjEGJfihnKodcAFOaeL
- $\Gamma^{(6)}$ DABCGEFJHIMKLONJMOEDNHGKAIFBLCKehObIkcmEgDiCLGJnljMANfBaoFHd
- $\Gamma^{(7)}$ DABCGEFJHIMKLONJMOEDNHGKAIFBLCKigOmIckbEhDeCLHFnljBNAfMaoJGd
- $\Gamma^{(8)}$ DABCGEFJHIMKLONJMOEDNHGKAIFBLCKJgOmjcAMfHDeCLhFnlIBNkEbaoiGd
- $\Gamma^{(9)}$ DABCGEFJHIMKLONJMOEDNHGKAIFBLCKFniMOAndmGDjhBgoHLJbkCLEaIecf

The first step, in order to distinguish and recognize the manifolds represented by $\Gamma^{(1)}, \dots, \Gamma^{(9)}$, has been to write a presentation of their fundamental groups by means of the algorithm described in [17].

By abelianizing the presentation, we could compute the first homology group of the involved manifolds. More precisely, we had

$$\begin{aligned}
H_1(|K(\Gamma^{(1)})|) &= H_1(|K(\Gamma^{(5)})|) = \mathbb{Z} \oplus \mathbb{Z}_2 \\
H_1(|K(\Gamma^{(2)})|) &= \mathbb{Z} \oplus \mathbb{Z}_3 \\
H_1(|K(\Gamma^{(3)})|) &= \mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \\
H_1(|K(\Gamma^{(4)})|) &= H_1(|K(\Gamma^{(9)})|) = \mathbb{Z} \\
H_1(|K(\Gamma^{(6)})|) &= H_1(|K(\Gamma^{(7)})|) = \mathbb{Z} \oplus \mathbb{Z} \\
H_1(|K(\Gamma^{(8)})|) &= \mathbb{Z} \oplus \mathbb{Z}_8.
\end{aligned}$$

The above list allows us to establish that $\Gamma^{(2)}, \Gamma^{(3)}, \Gamma^{(8)}$, represent distinct manifolds, which are distinct from all others.

With regard to $\Gamma^{(1)}, \Gamma^{(2)}, \Gamma^{(3)}$, however, a close analysis of the fundamental groups led us to their identification as follows.

Let us consider the following presentation of $\pi_1(|K(\Gamma^{(1)})|)$, which comes from the algorithm in [17] by choosing the colours $\{1, 3\}$:

$$< a, b, c, d, e / a^{-1}cd^{-1}, a^{-1}be, c^{-1}ed^{-1}b, c^{-1}da, e^{-1}ab^{-1}da^{-1} >$$

We get d from the first and fourth relation, e from the second relation and substitute them in the remaining ones; hence we obtain the new presentation

$$\langle a, b, c \mid [a^{-1}, c] = 1, bcb^{-1}ca^{-2}, ba^{-1}b^{-1}a^3c^{-1} \rangle$$

which is easily recognized as a presentation of $\mathbb{Z} \begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix}$, i.e. the semidirect product of \mathbb{Z} and $\mathbb{Z} \times \mathbb{Z}$ induced by the indicated matrix; hence $\pi_1(|K(\Gamma^{(1)})|) = \pi_1(TB \begin{pmatrix} 3 & 1 \\ -2 & -1 \end{pmatrix}) = \pi_1(TB \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix})$, since the last two matrices are conjugated in $GL(2, \mathbb{Z})$.

Analogously, if we consider the following presentation of $\pi_1(|K(\Gamma^{(2)})|)$ (the chosen colours are $\{0, 2\}$):

$$\langle a, b, c, d, e \mid a^{-1}bd^{-1}ec, c^{-1}db, d^{-1}ae^{-1}, a^{-1}ec^{-1}d, d^{-1}b^{-1}c \rangle$$

by using the second, fifth and third relations to obtain d and e and rewriting the remaining relations only in a, b, c , we have

$$\langle a, b, c \mid [b^{-1}, c] = 1, aca^{-1}b^3c^{-2}, ab^{-1}a^{-1}bc^{-1} \rangle$$

which is $\mathbb{Z} \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$. Again, we have that $|K(\Gamma^{(2)})|$ has the same fundamental group as the torus bundle $TB \begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix} = TB \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$.

Finally, we can perform similar transformations starting from

$$\pi_1(|K(\Gamma^{(3)})|) = \langle a, b, c, d, e \mid b^{-1}ca^{-1}e^2, d^{-1}ae, d^{-1}ea, a^{-1}ec^{-1}eb, a^{-1}bc^{-1} \rangle$$

and obtain

$$\pi_1(|K(\Gamma^{(3)})|) = \pi_1(TB \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}).$$

The former analysis of fundamental groups led us to check the suspected identifications; as a consequence we established the following

Proposition 4 $|K(\Gamma^{(1)})| = TB \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$

$$|K(\Gamma^{(2)})| = TB \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$$

$$|K(\Gamma^{(3)})| = TB \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$$

Proof. By means of the algorithm described in [9], we constructed triangulations of $TB \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$, $TB \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$, $TB \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ (see Figure 4.3 for the last) and, by means of dipoles

cancellation and switching of ρ -pairs³, we obtained rigid crystallizations of the above torus bundles, which were added to the set $cl(\Gamma^{(1)}) \cup cl(\Gamma^{(2)}) \cup cl(\Gamma^{(3)})$. The classification program applied to this list produced exactly three classes, proving the above identifications. ■

In order to analyze the remaining six unknown classes, we used a combinatorial invariant of 3-manifold, the *GM-complexity* which is an upper bound for Matveev's complexity ([24], [25]).

The GM-complexity $c_{GM}(M)$ of a 3-manifold M is defined as $\min c(\Gamma)$, where the minimum is taken among all crystallizations Γ of M and $c(\Gamma)$, the *complexity* of the 4-coloured graph Γ , is an integer which can be computed on Γ by means of an easily implemented algorithm (see [10] and [13] for precise definition and results).

By the computation of complexity performed on all representatives of $cl(\Gamma^{(4)}), \dots, cl(\Gamma^{(9)})$ ⁴, we had

$$\begin{aligned} c_{GM}(|K(\Gamma^{(i)})|) &\leq 8, \quad \text{for } i = 4, 5 \\ c_{GM}(|K(\Gamma^{(i)})|) &\leq 9, \quad \text{for } i = 6, 7, 8, 9. \end{aligned}$$

Table 9 of [7], which shows the closed non-orientable 3-manifolds up to Matveev's complexity 10, gave us possible identifications for $|K(\Gamma^{(i)})|$, $i = 4, \dots, 9$ and the program *Regina*, realized by B. Burton⁵ gave us the list of minimal triangulations of these manifolds.

It is easy to see that, given a triangulation T of a manifold M , a coloured triangulation of M can be always constructed by taking the first barycentric subdivision T' of T and labelling each vertex $v \in V(T')$ by the dimension of its dual simplex of T .

Moreover the 4-coloured graph dual to this coloured triangulation can be always reduced, by deleting dipoles and switching of ρ -pairs, to a rigid crystallization of M .

Therefore, Burton's triangulations allowed us to obtain a list X of crystallizations of manifolds, which were possible candidates for our identifications: the set $\bigcup_{i=4}^9 cl(\Gamma^{(i)}) \cup X$ was handled by Γ -class, whose results are summarized in the following proposition.

Proposition 5 *The classes $cl(\Gamma^{(4)}), \dots, cl(\Gamma^{(9)})$ all represent distinct manifolds, which are precisely:*

$$\begin{aligned} |K(\Gamma^{(4)})| &= SFS(\mathbb{RP}^2, (2, 1), (3, 1)) \\ |K(\Gamma^{(5)})| &= SFS(\overline{D}, (2, 1), (3, 1)) \\ |K(\Gamma^{(6)})| &= SFS(T^2/o_2, (2, 1)) \\ |K(\Gamma^{(7)})| &= SFS(K^2, (2, 1)) \\ |K(\Gamma^{(8)})| &= SFS(K^2/n_3, (2, 1)) \\ |K(\Gamma^{(9)})| &= SFS(A, (2, 1)) \cup SFS(A, (2, 1)) / \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \end{aligned}$$

³This process was performed by using program Duke III, available at WEB page <http://cdm.unimo.it/home/matematica/casali.mariarita/DukeIII.htm>, which manipulates edge-coloured graphs.

⁴Again GM-complexity computation was performed by a C++ program, *CGM* (authors M.R. Casali and P. Cristofori), available at WEB page <http://cdm.unimo.it/home/matematica/casali.mariarita/DukeIII.htm>

⁵the program *Regina* is available at WEB page <http://regina.sourceforge.net>

As a consequence of the above proposition and the results of the previous section, we can state

Proposition 6 *There exists a one-to-one correspondence between the set of classes of $\tilde{\mathcal{C}}^{(30)}$ produced by the classification program and the set of non-orientable 3-manifolds admitting a coloured triangulation with at most 30 tetrahedra.*

The following Table summarizes our results with regard to prime non-orientable manifolds admitting a coloured triangulation with at most 30 tetrahedra ⁶.

⁶We recall that $\mathbb{E}^3/\mathcal{B}_i$ with $i = 1, 2, 3, 4$ denote the four non-orientable Euclidean manifolds according to the notations of [31].

tetrahedra	3-manifold
14	$\mathbb{S}^1 \times \mathbb{S}^2$
16	$\mathbb{RP}^2 \times \mathbb{S}^1$
24	$\mathbb{E}^3/\mathcal{B}_1$ $\mathbb{E}^3/\mathcal{B}_2$ $\mathbb{E}^3/\mathcal{B}_4$
26	$\mathbb{E}^3/\mathcal{B}_3$ $TB \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$
28	$TB \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}$ $SFS(\mathbb{RP}^2; (2, 1), (3, 1))$
30	$TB \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$ $SFS(\overline{D}; (2, 1), (3, 1))$ $TB \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$ $SFS(T^2/o_2; (2, 1))$ $SFS(K^2; (2, 1))$ $SFS(K^2/n_3; (2, 1))$ $SFS(A; (2, 1)) \cup SFS(A; (2, 1))/ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

TABLE 2: Prime 3-manifolds represented by crystallizations of $\tilde{C}^{(30)}$

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